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Neighbourhood Polynomials Derived Through Binary Operations on Graphs

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Abstract

Binary operations on graphs are studied widely in graph theory ever since each of these operations has been introduced. The neighbourhood polynomial plays a vital role in describing the neighbourhood characteristics of the vertices of a graph. In this study neighbourhood polynomial of graphs arising from the operations like conjunction, join and symmetric difference of certain classes of graphs are calculated and tried to characterize the nature of neighbourhood polynomial.

Key words: Conjunction, Join, Symmetric difference Neighbourhood Polynomial

Introduction

The neighbourhood polynomials of the graphs resulting from Cartesian product have been studied and some properties have been established in [3].

1.1. The operations on graphs in this study

The operation of conjunction ($\wedge$) on graphs was introduced by Weichsel in 1963. For any two graphsG_1 and G_2, it is denoted as $G = G_1 \wedge G_2$ and is defined as $V(G) = V(G_1) \times V(G_2)$, two vertices (u_i, v_j), (u_k, v_l) are adjacent if u_i adjacent to u_k inG_1 and $v_j$adjacent to v_l in G_2. Join of two graphs G_1 and G_2 is denoted as $G = G_1 \vee G_2$. In join, $V(G) = V(G_1) \cup V(G_2)$, edge set consists of edges of G_1 and G_2 together with all edges joining every vertex of G_1 to every vertices of G_2. The symmetric difference ($\oplus$) between any two graphs G_1 and G_2, it is denoted as $G = G_1 \oplus G_2$ and is defined as $V(G) = V(G_1) \times V(G_2)$, two vertices (u_i, v_j), (u_k, v_l) are adjacent if either u_i adjacent to u_k inG_1 or v_j adjacent to v_l in G_2, but not the both. For notations and terminology we follow [2].

1.2. Neighbourhood complex and polynomial

A complex on a finite set $\mathcal{X}$ is a collection $\mathcal{C}$ of subsets of$\mathcal{X}$, closed under certain predefined restriction. Each set in $\mathcal{C}$ is called the face of the complex. In the neighbourhood complex $\mathcal{N}(G)$ of a graph$G$, $\mathcal{X} = V(G)$, and faces are subsets of vertices that have a common neighbour. In [1] the neighbourhood polynomial of a graph G, is defined as

$neigh_G(x) = \sum_{u\in\mathcal{N}(G)} x^{|u|}$.

For example consider C_4 with vertices $\{a,b,c,d\}$. The neighbourhood complex $\mathcal{N}(C_4)$ of C_4 is $\{\phi,\ \{a\},\ \{b\},\ \{c\},\ \{d\},\ \{a,c\},\ \{b,d\}\}$Since the empty set trivially has a common neighbour, each of the single vertices has a neighbour, the sets $\{a,c\},\{b,d\}$ has two common neighbours (one is sufficient), but no three vertices have a common neighbour. The associated neighbourhood polynomial of $C_4$is $neigh_{C_4}(x) = 1 + 4x + 2x^2$.

Similarly, the neighbourhood polynomials of certain standard graphs are as follows:

1. K_n - $neigh_{K_n}(x) = (1+x)^n - x^n$.
2. P_n - $neigh_{P_n}(x) = 1 + nx + (n-2)x^2$.
3. $C_n - neigh_{C_4}(x) = \begin{cases} 1 + nx + nx^2, n \neq 4 \\ 1 + nx + 2x^2, n = 4 \end{cases}$.

In this paper, neighbourhood polynomials for the graphs resulting from the binary operations of conjunction, join, and symmetric difference are calculated. Also tried to characterize some properties of the neighbourhood polynomial of the graph G so formed.

2. Main Results

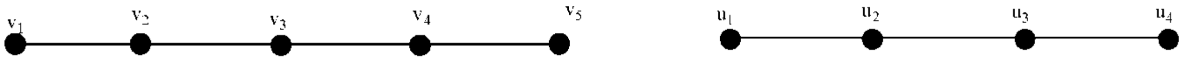
2.1 Conjunction of two graphs and their Neighbourhood Polynomials

Lemma 2.1.1 The neighbourhood polynomial of mesh graph is

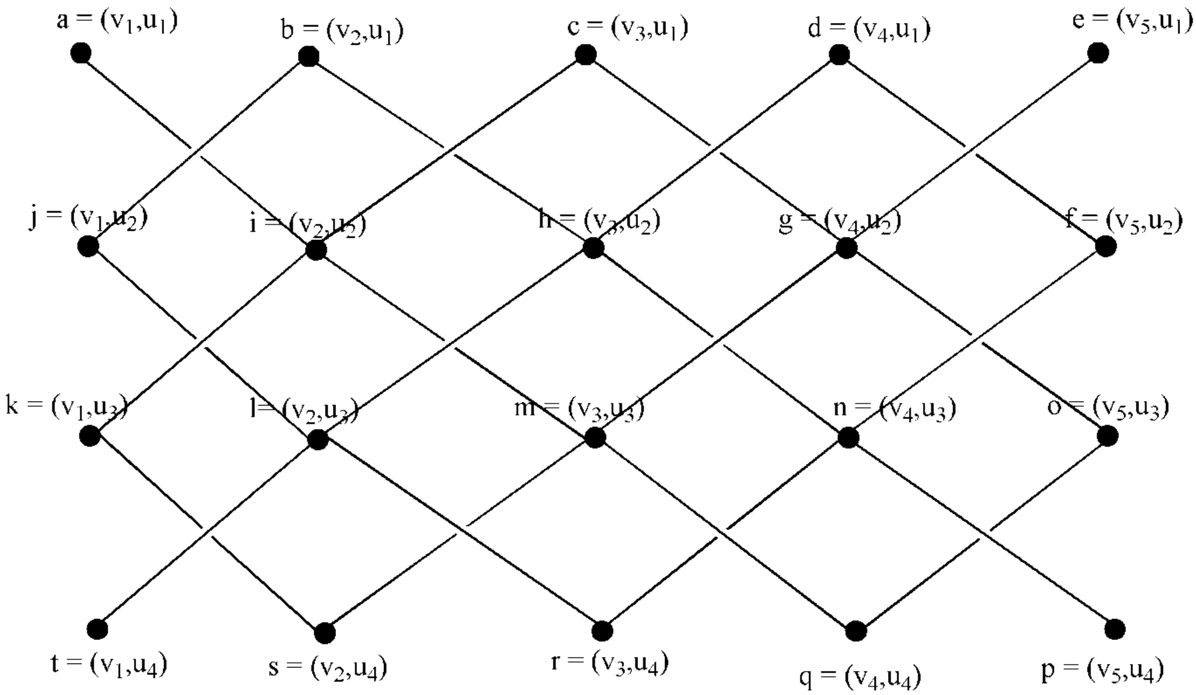
$1 + mnx + [4mn - 6(m+n) + 8]x^2 + (m-2)(n-2)(4x^3 + x^4)$.

Proof.Consider the mesh graph $G = P_n \wedge P_m$. In $P_n \wedge P_m$ there are mn vertices. The empty set trivially has a neighbour and each of the mn single vertices has a neighbour.

Now consider the figure 1, $P_5 \wedge P_4$



P_5 P_4



$G = P_5 \wedge P_4$

Figure 1

The two element subsets $\{\{a,k\},$ [illegible] $s\}, \{c,m\}, \{h,r\}, \{d,x\}, \{g,q\},\{e,o\}, \{f,p\}\}[m(n-2)=5(4-2)=10]$; $\{\{a,c\}, \{b,d\}, \{c,e\}, \{j,h\}, \{i,g\}, \{h, f\},\{k,m\}, \{l,n\}, \{m,o\}, \{t,r\}, \{s,q\}, \{r,p\}\}[n(m-2)=4(5-2)=12]$; and $\{\{j,r\}, \{a,m\}, \{i,q\}, \{b,n\}, \{h,p\}, \{c,o\}, \{c,k\}, \{d,l\}, \{h,t\}, \{e,m\}, \{g,s\}, \{f,r\}\}[2(m-2)(n-2)]$; have at least one common neighbour. The three element subsets having at least one common neighbour are $\{\{c,e,m\}, \{c,e,o\}, \{c,m,o\}, \{e,m,o\}, \{b,d,l\}, \{b,d,n\}, \{b,l,n\}, \{d,l,n\}, \{a,c,k\}, \{a,c,m\}, \{a,k,m\}, \{c,k,m\}, \{h,j,r\}, \{h,j,t\}, \{h,r,t\}, \{j,r,t\}, \{g,i,q\}, \{g,i,s\}, \{g,q,s\}, \{i,q,s\}, \{f,h,p\}, \{f,h,r\}, \{f,p,r\}, \{h,p,r\}\}[4(m-2)(n-2)=4(5-2)(4-2)=24]$ and $\{\{c,e,m,o\}, \{b,d,l,n\}, \{a,c,k,m\}, \{h,j,r,t\}, \{g,i,q,s\}, \{f,h,p,r\}\}[(m-2)(n-2)=(5-2)(4-2)=6]$ are the four element subsets having at least one common neighbour.

Thus for $G = P_5 \wedge P_4$, the neighbourhood polynomial is

$neigh_G(x) = 1 + 20x + 34x^2 + 24x^3 + 6x^4$.

Generally, for $G = P_m \wedge P_n$,

$neigh_G(x) = 1 + mnx + [4mn - 6(m+n) + 8]x^2 + (m-2)(n-2)(4x^3 + x^4)$.

Corollary 2.1.2 The neighbourhood polynomial of $P_m \wedge K_2$is $1 + 2mx + (2m - 4)x^2$.

Proof. We have,
$neigh_{P_m \times P_n}(x) = 1 + mnx + [4mn - 6(m + n) + 8]x^2 + (m - 2)(n - 2)(4x^3 + x^4)$.
When $n = 2$, we get, $neigh_{P_m \times K_2}(x) = 1 + 2mx + (2m - 4)x^2$.

Lemma 2.1.3 The neighbourhood polynomial of $C_m \wedge C_n$ is,
$1 + mnx + 4mn(x^2 + x^3) + mnx^4, m \neq n \neq 4$.

Proof. Consider, $G = C_m \wedge C_n, m \neq n \neq 4$. From the definition of conjunction, for every $v_j \in V(G)$, we have $d(v_j) = 4$. That is, there corresponds 4 neighbours to every vertex v_j of G
To find set of vertices having at least one common neighbour, say v_j, we compute, $\binom{4}{2}, \binom{4}{3}, \binom{4}{4}$, of the four neighbouring vertices of v_j. Since in G, there are mn vertices, in the neighbourhood complex of G we have null set, mn single vertices, $mn\binom{4}{2} = 6mn$, two element subsets, $4mn$ three element subsets and $4mn$ four element subsets.
On considering $C_m \wedge C_n$, for different m and n, it is verified that there are only
$(6mn - 2mn) = 4mn$distinct two element subsets of vertices having at least a common neighbour.
Hence, $neigh_G(x) = 1 + mnx + 4mn(x^2 + x^3) + mnx^4, m \neq n \neq 4$.

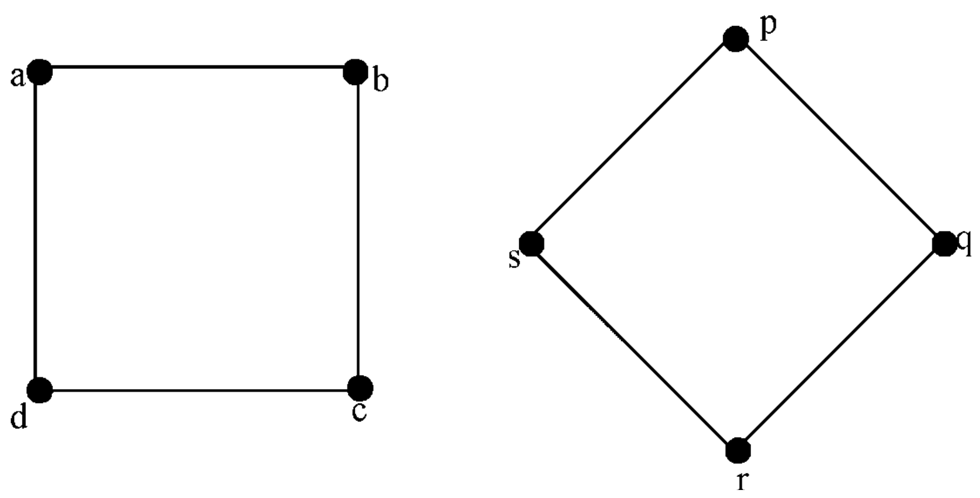
Corollary 2.1.4 The neighbourhood polynomial of $C_m \wedge C_4$ is,
$1 + 4mx + 10mx^2 + 8mx^3 + 2mx^4, m \neq 4$.

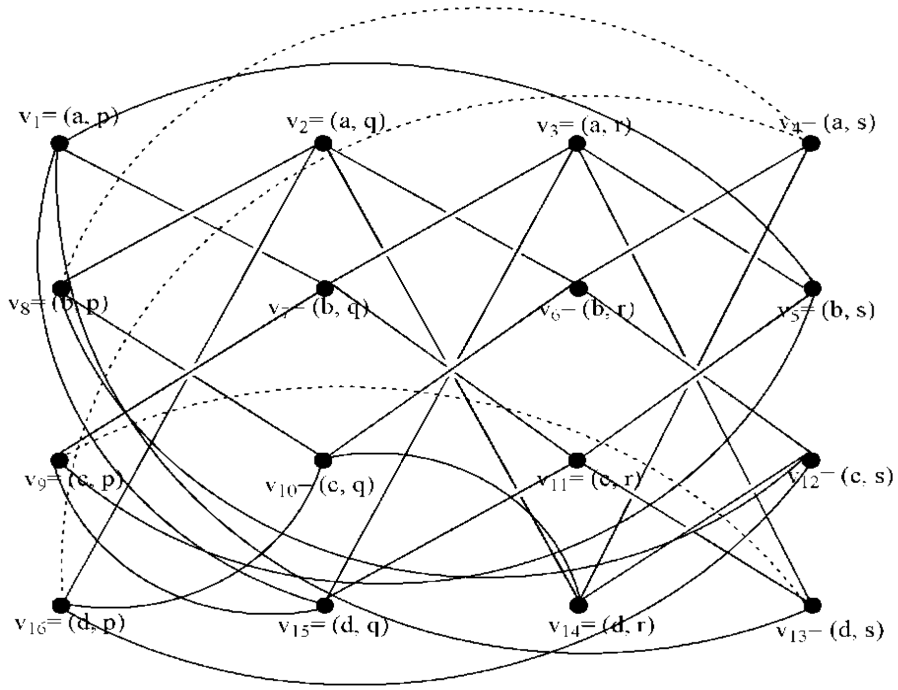
Proof.Let $G = C_m \wedge C_n$. $|V(G)| = mn$. Each of the mn vertices has 4 neighbours. When $n = 4$, the neighbours of first $mn/2$ vertices is same as that of later $mn/2$ vertices. That is, we have to consider the neighbours of only $4m/2 = 2m$, vertices are only needed to be considered(since, we are finding the distinct set of vertices having common neighbours). Following the same argument as in lemma 2.1.3, we get

$neigh_{C_m \wedge C_4}(x) = 1 + 4mx + 10mx^2 + 8mx^3 + 2mx^4, m \neq 4$.

Remark. The neighbourhood polynomial of $C_4 \wedge C_4$ is$1 + 16x + 24x^2 + 16x^3 + 4x^4$.

Consider figure 2,$G = C_4 \wedge C_4$





$G = C_4 \wedge C_4$

Figure 2

Here, the each vertex of the set $\{v_1, v_3, v_9, v_{11}\}$ have same set of neighbours as that of $\{v_2, v_4, v_{10}, v_{12}\}$ and vice versa. Also for the vertices $\{v_5, v_7, v_{13}, v_{15}\}$ and $\{v_6, v_8, v_{14}, v_{16}\}$.

The neighbourhood polynomial is is$1 + 16x + 24x^2 + 16x^3 + 4x^4$.

Lemma 2.1.5 The neighbourhood polynomial of $P_m \wedge C_n$ is

$$1 + mnx + (4mn - 6n)x^2 + 4n(m-2)x^3 + n(m-2)x^4, n \neq 4.$$

Proof. Let $G = P_m \wedge C_n$. Ghasmn vertices, 2 vertices of P_m is of degree 1 and $(m-2)$ vertices of P_m, and n vertices of C_n are of degree 2. Hence in $G = P_m \wedge C_n$, $2n$ vertices are of degree 2, and $(m-2)n$ vertices are of degree 4. The neighbourhood complex of G consists of null vertex along with mn single vertices. The number of two element simplexes are $(n-2)m + (m-2)n + 2m + 2n(m-2) = (4mn - 6n)$, the three element simplexes count to $4n(m-2)$and there are $n(m-2)$ four element simplexes. Also there is no set of five more vertices having a common neighbour in $P_m \wedge C_n$.

Hence the neighbourhood polynomial of $P_m \wedge C_n$ is,

$$neigh_{P_m \wedge C_n}(x) = 1 + mnx + (4mn - 6n)x^2 + 4n(m-2)x^3 + n(m-2)x^4, n \neq 4.$$

Corollary 2.1.6 The neighbourhood polynomial of $P_m \wedge C_4$is,
$1 + 4mx + (10m - 16)x^2 + 8(m-2)x^3 + 2(m-2)x^4$.

Proof. Let $G = P_m \wedge C_4$. Then G has $4m$ vertices, of which 8 vertices are of degree 2 and $(4m-8)$ vertices are of degree 4.In $P_m \wedge C_n$, there are $(n-2)m + (m-2)n + 2m + 2n(m-2)$, two element subsets of vertices having at least a common neighbour. When $n = 4$, first subset of $n(m-2)$ two element vertices coincides with later $n(m-2)$ two element subsets of vertices and $2m$ subsets with two elements coincides with $n(m-2)$ subsets of vertices.

Thus we have,

$$(4mn - 6n) - n(m-2) - 2m = 3mn - 4n - 2m$$
$$= 10m - 16(\ since\ n = 4),$$

two simplexes. Also when $n = 4$, the neighbours of first $2m$ set of vertices are same as that of later $2m$ set of vertices. Hence the number of three and four element subsets are $8(m-2)$ and $2(m-2)$ respectively.

Thus for $G = P_m \wedge C_4$,

$$neigh_G(x) = 1 + 4mx + (10m - 16)x^2 + 8(m-2)x^3 + 2(m-2)x^4.$$

Theorem 2.1.7 If $G = G_1 \wedge G_2$, then , $deg\big(neigh_G(x)\big) = \Delta(G_1) \times \Delta(G_2)$.

Proof. Let $\{u_1, u_2, u_3, \dots, u_m\} \in V(G_1)$ and $\{v_1, v_2, v, \dots, v_n\} \in V(G_2)$. For any vertex, $w_i = (u_k, v_j)$, in G,

$d(w_i) = d(u_k) \times d(v_j)$, which follows from the definition of $G_1 \wedge G_2$.

$d(w_i)$ is maximum, only if $d(u_k) = \Delta(G_1)\ and\ d(v_j) = \Delta(G_2)$. Consider the neighbourhood complex $\mathcal{N}(G)$ of G. The $d(w_i)$, vertices adjacent to w_i, forms complexes with one element, two elements, three elements, …, $d(w_i)$ elements (since, these $d(w_i)$ vertices have at least a common neighbour w_i) and also no $[d(w_i) + 1]$ vertices can have w_i as a common neighbour. Thus in G, there exists a maximal face with respect to a vertex with maximum degree.

Also we have, $neigh_G(x) = \sum_{u \in \mathcal{N}(G)} x^{|u|}$, which implies, $deg\big(neigh_G(x)\big)$, is the maximum cardinality of the face in the neighbourhood complex. Thus if $w_i \in V(G)$, with

$d(w_i) = \Delta(G_1) \times \Delta(G_2)$,

$deg\big(neigh_G(x)\big) = \Delta(G_1) \times \Delta(G_2)$.

2.2 Join of two graphs and their Neighbourhood Polynomials.

Lemma 2.2.1 The neighbourhood polynomial of fan graph F_n is

$1 + (n+1)x + \left(\binom{n}{2} + n\right)x^2 + \left[\binom{n}{3} + (n-2)\right]x^3 + \binom{n}{4}x^4 + \cdots + x^n$.

Proof. The fan graph $F_n = P_n \vee K_1$. F_n consists of P_n, along with edges joining every vertex $v_i, i = 1,2, \dots n$, of P_n, to the single vertex u of K_1. Thus F_n has $(n+1)$ vertices.

The neighbourhood complex $\mathcal{N}(F_n)$, of F_n is,

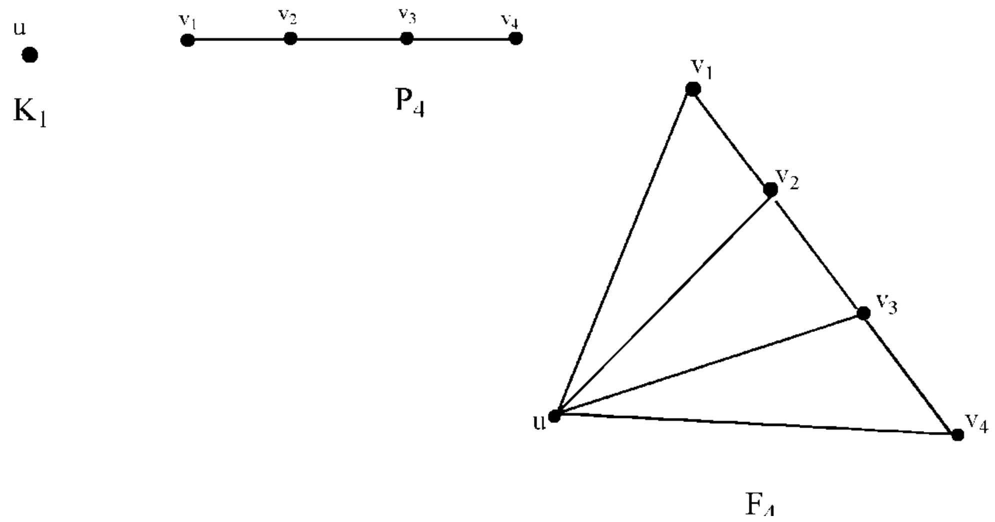
$\mathcal{N}(F_n) = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \dots, \{v_n\}, \{u\}, \{v_1, v_2\}, \{v_1, v_3\}, \dots, \{v_1, v_n\}, \{v_2, v_3\}, \{v_2, v_4\}, \dots,$
$\{v_2, v_n\}, \dots, \{v_{n-1}, v_n\}, \{v_1, u\}, \{v_2, u\}, \dots, \{v_n, u\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \dots, \{v_1, v_2, v_n\}, \dots,$
$\{v_{n-2}, v_{n-1}, v_n\}, \{v_1, v_3, u\}, \{v_2, v_4, u\}, \dots, \{v_{n-2}, v_n, u\}, \{v_1, v_2, v_3, v_4\}, \dots,$
$\{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}, \dots, \{v_1, v_2, v_3, \dots, v_n\}\}$.

From the neighbourhood complex of F_n we get,

$$neigh_{F_n}(x) = 1 + (n+1)x + \left(\binom{n}{2} + n\right)x^2 + \left[\binom{n}{3} + (n-2)\right]x^3 + \binom{n}{4}x^4 + \cdots + x^n.$$

Example

Consider $F_4 = P_4 \vee K_1$,



F_4

Figure 3

$$\mathcal{N}(F_n) = \Big\{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{u\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, u\}, \{v_2, u\}, \{v_3, u\}, \{v_4, u\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}, \{v_1, u, v_3\}, \{v_2, u, v_4\}, \{v_1, v_2, v_{3,} v_4\}\Big\}$$

From the definition of neighbourhood polynomial we have $neigh_{F_n}(x) = \sum_{u \in \mathcal{N}(F_n)} x^{|u|}$. Hence, $neigh_{F_4}(x) = 1 + 5x + 10x^2 + 6x^3 + x^4$.

Lemma 2.2.2 The neighbourhood polynomial of W_n is

$$1 + (n+1)x + \binom{n}{2} + n\Big) x^2 + \left(\binom{n}{3} + n\right) x^3 + \binom{n}{4} x^4 + \cdots + x^n, n > 3.$$

Proof. We have $W_n = C_n \vee K_1$. Let $(v_1, v_2, v_3, \ldots, v_n) \in V(C_n)$and $V(K_1) = u$. In W_n, one vertex of the $(n+1)$ vertices, has n neighbours and others has three neighbours each.

The neighbourhood complex $\mathcal{N}(W_n)$ of W_n is,

$\mathcal{N}(W_n) = \{\varphi, \{v_1\}, \{v_2\}, \{v_3\}, \ldots, \{v_n\}, \{v_1, u\}, \{v_2, u\}, \ldots, \{v_{n-1}, v_n\}, \ldots, \{v_1, v_2, v_3, \ldots, v_n\}\}$.

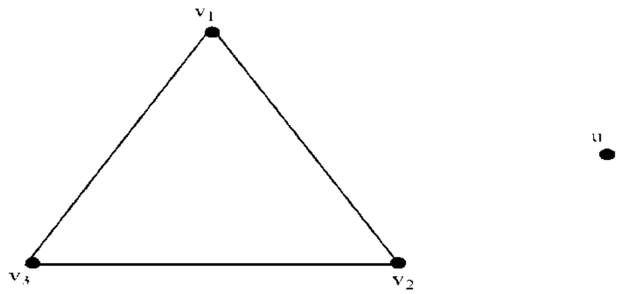
That is, the neighbourhood complex consists of empty set, which trivially having a common neighbour and subsets of vertices with *one* element, *two* elements, *three* elements, etc. up to n elements, with cardinalities $(n+1), \left(\binom{n}{2} + n\right), \left(\binom{n}{3} + n\right), \binom{n}{4}, \ldots, 1 (= \binom{n}{n})$, respectively.

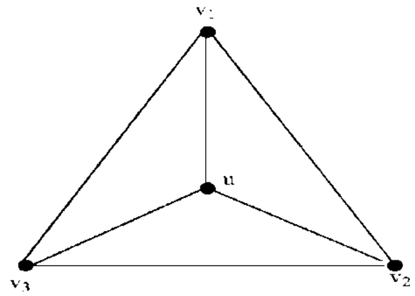
Hence, the neighbourhood polynomial of W_n is,

$$neigh_{W_n}(x) = 1 + (n+1)x + \binom{n}{2} + n\Big) x^2 + \left(\binom{n}{3} + n\right) x^3 + \binom{n}{4} x^4 + \cdots + x^n, n > 3.$$

Example

Consider $W_3 = C_3 \vee K_1$,





W_3

Figure 4

$\mathcal{N}(W_3) =$

$\{\varphi,\ \{v_1\},\ \{v_2\},\ \{v_3\},\ \{v_1,v_2\},\ \{v_1,v_3\},\ \{v_2,v_3\},\ \{v_1,u\},\ \{v_2,u\},\ \{v_3,u\},\ \{v_1,v_2,v_3\},$
$\{v_1,v_2,u\},\{v_1,v_3,u\},\ \{v_2,v_3,u\}\}.$

$neigh_{W_3}(x) = 1 + 4x + 6x^2 + 4x^3.$

Lemma 2.2.3 Let G_1 be a $r - regular$ graph and G_2 be a $s - regular$ graph of orders m and n respectively. Then $G = G_1 \vee G_2$ is regular if and only if, $r + n = s + m$.

Proof. Assume G is regular. Let $u_1, u_2, u_3, \dots, u_m \in V(G_1)$and $v_1, v_2, v_3, \dots, v_n \in V(G_2)$. In $G = G_1 \vee G_2$, each vertex u_i of G_1 is joined to every vertex of v_j of G_2, in addition to the edges of G_1 and G_2. Also since G_1 and G_2 are $r - regular$ and $s - regular$ respectively, every vertex u_i and v_j of G are of degree $r + n$and $s + m$, respectively. Since G is regular $r + n = s + m$.

Conversely assume, $r + n = s + m$.

$\Rightarrow deg(u_i) + n = deg(v_j) + m$, since G_1 is $r - regular$ and G_2 is $s - regular$

$\Rightarrow degree\ of\ any\ vertex\ u\ of\ G = degree\ of\ any\ vertex\ v\ of\ G.$

$\Rightarrow G\ is\ regular.$

Theorem 2.2.4 Let G_1 and G_2be any two graphs of order m and n respectively.

If $G = G_1 \vee G_2$ is a $s - regular$ graph, then,

$neigh_G(x) = 1 + (m+n)x + \left(\binom{m}{2} + mn + \binom{n}{2}\right)x^2 + \left(\binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}\right)x^3 + \left(\binom{m}{4} + \binom{m}{3}\binom{n}{1} + \binom{m}{2}\binom{n}{2} + \binom{m}{1}\binom{n}{3} + \binom{n}{4}\right)x^4 + \cdots + \left(\binom{m}{s} + \binom{m}{s-1}\binom{n}{1} + \cdots + \binom{m}{1}\binom{n}{s-1} + \binom{n}{s}\right)x^s.$

Proof. Since, G_1 and G_2 are any two graphs of order m and n respectively, in $G = G_1 \vee G_2$, there are $m+n$ vertices, such that every vertex of G_1 is joined to every vertex of G_2 through an edge, in addition to the edges of G_1 and G_2. Thus for every $u_i \in V(G)$, u_i has n more neighbours in addition to that which u_i has in G_1 and for every $v_j \in V(G)$, v_j has m more neighbours in addition to that which v_j has in G_2.

By definition the neighbourhood complex of Gconsists of the null set, $(m+n)$ single vertices, since each has a neighbour. Also since $G = G_1 \vee G_2$, any two vertices either in G_1 or in G_2 has a common neighbour, also any combination of u_i and v_j has a common neighbour. Thus the number of two element simplexes are $\left(\binom{m}{2} + mn + \binom{n}{2}\right)$.

On considering the number of simplexes with three elements, any 3 vertices of both G_1 and G_2 has a common neighbour, any 2 vertices of G_1 and any 1 vertex of G_2 has a common neighbour. Similarly any 1 vertex of G_1 and any 2 vertices of G_2 has a common neighbour. Thus there exists $\left(\binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}\right) 3 - simplexes.$

Similarly, the number of four simplexes are $\left(\binom{m}{4} + \binom{m}{3}\binom{n}{1} + \binom{m}{2}\binom{n}{2} + \binom{m}{1}\binom{n}{3} + \binom{n}{4}\right)$, since any 4 vertices of both G_1 and G_2 has a common neighbour, any 3 vertices of either G_1 or G_2 and any 1 vertex of either G_2 or G_1 has a common neighbour any two vertices of G_1 any two vertices of G_2 also have a common neighbour, for $G = G_1 \vee G_2$ is a regular graph.

The argument continues for all simplexes of length $s = deg(G)$.

Hence the neighbourhood polynomial of $G = G_1 \vee G_2$ is,

$$\begin{aligned} neigh_G(x) = 1 + (m+n)x &+ \left(\binom{m}{2} + mn + \binom{n}{2}\right)x^2 \\ &+ \left(\binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}\right)x^3 \\ &+ \left(\binom{m}{4} + \binom{m}{3}\binom{n}{1} + \binom{m}{2}\binom{n}{2} + \binom{m}{1}\binom{n}{3} + \binom{n}{4}\right)x^4 + \cdots \\ &+ \left(\binom{m}{s} + \binom{m}{s-1}\binom{n}{1} + \cdots + \binom{m}{1}\binom{n}{s-1} + \binom{n}{s}\right)x^s. \end{aligned}$$

Theorem 2.2.5 The neighbourhood polynomial of $K_m \vee K_n$ is of degree $m+n-1$.

Proof. Let $G = K_m \vee K_n$. In K_m, every vertex is of degree $(m-1)$ and that in K_n is $(n-1)$. Also these m vertices of K_m are joined to every n vertices of K_n. Hence in G the degree of each vertex belonging to K_m is $(m-1+n)$ and that belonging to $K_n$is $(n-1+m)$. Thus G is $(m+n-1)$ regular graph of order $(m+n)$. Thus the neighbourhood complex of G consists of the simplexes as described in the theorem 2.19, and since the maximum degree of $G$is $(m+n-1)$, no set of $(m+n)$ vertices have a common neighbour, the maximal simplex is $m+n-1$. Hence the $\deg(neigh_{K_m \vee K_n})$ is $m+n-1$.

Remark

It follows from the observations and theorems that, if $G = G_1 \vee G_2$where G_1 and $G_2$are any two graphs of order m and n respectively,

$max(m+2, n+2) \leq \deg(neigh_G(x)) \leq m+n-1.$

2.3 Symmetric difference of two graphs and their Neighbourhood Polynomials.

Theorem 2.3.1 The $deg(neigh_G(x)) = m$, where G is the symmetric difference of any graph G_1 of order m and K_2.

Proof. Let $G = G_1 \oplus K_2$. Then following the definition of symmetric difference of any two graphs G_1 and G_2, of orders m and n respectively, the degree of any vertex $u = (u_i, v_j)$ (where $u_i \in V(G_1)$ *and* $v_j \in V(G_2)$) in G is,

$$deg(u) = n \times deg(u_i) + m \times deg(v_j) - 2deg(u_i) \times deg(v_j).$$

Hence if $G = G_1 \oplus K_2$, for any vertex, $w = (u_i, v_j)$ in G, we have,

$deg(w) = 2 \times deg(u_i) + m \times 1 - 2 \times deg(u_i) \times 1$. (Since,$v_j \in K_2$, $deg(v_j) = 1$)

Thus $(w) = m$.

Hence on considering the neighbourhood complex $\mathcal{N}(G)$ of G, there exists no simplex of length $(m+1)$, as every vertex is of degree m, there exists simplexes of length $1, 2, 3, \dots, m$. Since, $neigh_G(x) = \sum_{u \in \mathcal{N}(G)} x^{|u|}$, the degree of $neigh_G(x)$ is equal to the length of maximal simplex. Hence, $deg(neigh_G(x)) = m$, where $G = G_1 \oplus K_2$.

Theorem 2.3.2 The $\left(neigh_G(x)\right) = m + n - 2$, if $G = K_m \oplus K_n$.

Proof. Let $G = K_m \oplus K_n$. Then degree of any vertex $w = \left(u_i, v_j\right)$(where $u_i \in V(K_m)$ *and* $v_j \in V(K_n)$) in G is,

$$\begin{aligned} deg(w) &= (m-1)n + (n-1)m - 2(m-1)(n-1) \\ &= m + n - 2. \end{aligned}$$

Also, we have $neigh_G(x) = \sum_{u \in \mathcal{N}(G)} x^{|u|}$. The elements of the neighbourhood complex $\mathcal{N}(G)$ of G, consists of the zero simplex, mn - single vertices as each has a neighbour, $2-$ simplexes, $3-$ simplexes, etc. to $(m+n-2)-$simplexes and there exists no simplex of length $(m+n-1)$ or more. Hence the degree of neighbourhood polynomial of $G = K_m \oplus K_n$, is $(m+n-2)$.

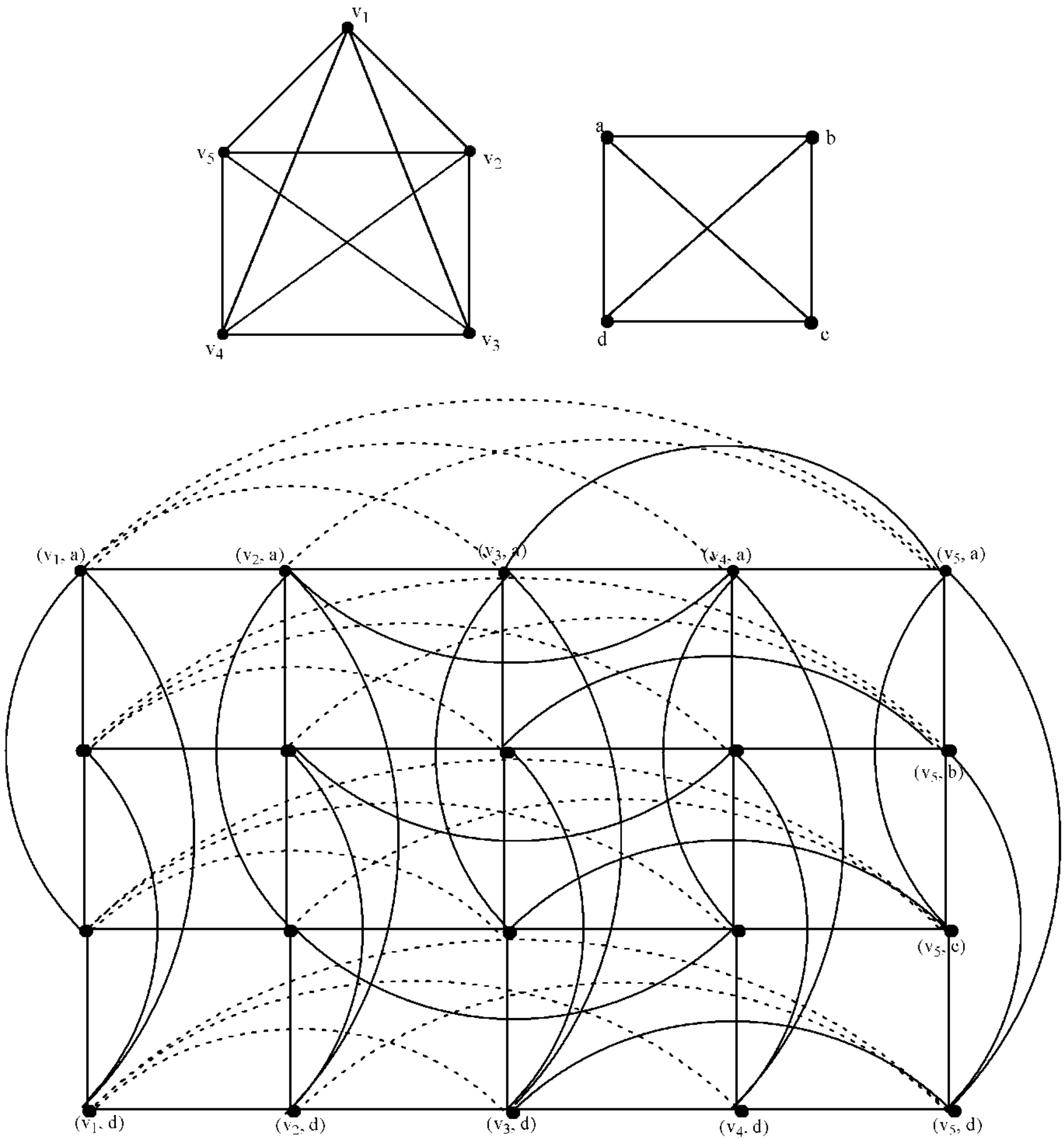
Theorem 2.3.3 If $G = K_m \oplus K_n$, then $neigh_G(x) = 1 + (mn)x + \binom{mn}{2}x^2 + \left[n\binom{m}{3} + n\binom{m}{2}(m-2)(n-1) + m\binom{n}{2}(n-2)(m-1) + m\binom{n}{3}\right]x^3 + \cdots + mn\binom{s}{i}x^i + \cdots + mnx^s$, $s = m+n-2$, $s/2 \le i \le s$.

Proof. $G = K_m \oplus K_n$, has mn vertices, each of these vertices have $(m+n-2)$ neighbours, (which follows from the definition of symmetric difference of two graphs). The neighbourhood complex of G consists of zero simplex, $1-simplexes$, since each of the mn vertices has a neighbour. Any two of mn vertices in $G = K_m \oplus K_n$ has a common neighbour, for consider vertices $\left(u_i, v_j\right)$ and (u_k, v_l) of G, where $u_i \in V(K_m)$ and $v_j \in V(K_n)$. Then there exists at least one vertex (u_i, v_l) of G which is common to both $\left(u_i, v_j\right)$ and (u_k, v_l), by the definition of $K_m \oplus K_n$. Thus the number of two element simplexes in the neighbourhood complex of G are $\binom{mn}{2}$. The three element simplexes are calculated as $n\binom{m}{3} + n\binom{m}{2}(m-2)(n-1) + m\binom{n}{2}(n-2)(m-1) + m\binom{n}{3}$ (taking $m, n > 3$). Continuing the same process, we get $i-simplexes$ to be $mn\binom{s}{i}$, where $s = m+n-2$ and $s/2 \le i \le s$, and since the maximal simplex of $G = K_m \oplus K_n$, is of length $m+n-2$, as there are $mn-simplexes$ of length $m+n-2$. Thus we get

$$\begin{aligned} neigh_G(x) = 1 + (mn)x + \binom{mn}{2}x^2 \\ + \left[n\binom{m}{3} + n\binom{m}{2}(m-2)(n-1) + m\binom{n}{2}(n-2)(m-1) + m\binom{n}{3}\right]x^3 \\ + \cdots + mn\binom{s}{i}x^i + \cdots + mnx^s, \qquad s = m+n-2, \qquad s/2 \le i \le s. \end{aligned}$$

Example

Consider figure 5, $G = K_5 \oplus K_4$



$G = K_5 \oplus K_4$

Figure 5

The neighbourhood complex of G consists of the null simplex, $20, 1 - simplexes$ of single vertex. Every pair of vertices arbitrarily taken has a common neighbour, consider the vertices (v_1, a) and (v_5, c) which has a common neighbour (v_1, c). Thus there are $\binom{20}{2} = 190$ two simplexes. Considering the neighbours of each vertex and finding out the possible

$3-simplexes$, and on cancelling the repetitions we get the number of $3-simplexes$, in $K_5 \oplus K_4$ to be 660 .(In $K_5 \oplus K_4$ each vertex has $5+4-2=7$ neighbours and ${}^{7}/{}_{2}=3.5$).

There are $20 \times \binom{7}{4} = 700$, $4-simplexes$, $20 \times \binom{7}{5} = 420, 5-simplexes$, $20 \times \binom{7}{6} = 140, 6-simplexes$ and $7-simplexes$ count to 20, for the simplexes $i = 4,5,6,7$, $i > {}^{7}/{}_{2}$, and there is no repetition of the same simplex. Thus,

$$neigh_G(x) = 1 + 20x + 190x^2 + 660x^3 + 700x^4 + 420x^5 + 140x^6 + 20x^7.$$

3.Conclusion and further scope

The neighbourhood polynomials on different binary operations on graphs are obtained and neighbourhood polynomials of other binary operations on graphs are still to be obtained

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